Mathematics 1210 PRACTICE EXAM II Fall 2004

ANSWER KEY

1. a. Using the chain rule, $x''(t) = -\omega^2 A \sin(\omega t - \phi) = -\omega^2 x(t)$

   b. Using the chain rule and the quotient rule, $f'(x) = 3\left(\frac{x-2}{x-\pi}\right)^2 \frac{(2-\pi)}{(x-\pi)^2}$

   c. Implicit differentiation treats y as a function of x even though it is not known explicitly. First differentiate the equation defining the relationship between x and y with respect to x and apply the usual differentiation rules (chain rule and product rule in the first term) to obtain $-\sin(xy)(y + xy') = 2yy' + 2$. Then collect terms which include a factor of y' from the chain rule and factor it out. $-y\sin(xy) - 2 = (2y + x\sin(xy))y'$ Finally, divide to solve for y': $y' = -(2 + y\sin xy)/(2y + x\sin xy)$

   d. Multiply the top and bottom by $1 + \cos x$ to get

   $$\lim_{x\to 0} \frac{\sin x \tan x}{1 - \cos x} = \lim_{x\to 0} \frac{(1+\cos x)}{\cos x} = 2.$$

   (or use $\sin x \sim x$, $\tan x \sim x$, $\cos x \sim 1 - x^2/2$ as $x \to 0$)

   e. This is of the form $f(g(\frac{u(x)}{v(x)}))$, where $f(x) = \sin x$, $g(x) = x^{1/2}$, $u(x) = \tan x$, and $v(x) = 1 + x^2$, so using the chain rule (twice) and quotient rule, $f'(x) = \cos\sqrt{\frac{\tan x}{1+x^2}}\ \frac{1}{2}\left(\frac{\tan x}{1+x^2}\right)^{-1/2} \frac{(1+x^2)\sec^2 x - 2x\tan x}{(1+x^2)^2}$

   f. This is of the form $p(x)q(r(s(x)))$ where $p(x) = x^2$, $q(x) = x^2$, $r(x) = \sin x$, $s(x) = x^3$, so using the product rule and the chain rule (twice) $f'(x) = 2x\sin^2(x^3) + 6x^4\sin(x^3)\cos(x^3)$. I like to rewrite $\sin^2(x^3)$ as $(\sin(x^3))^2$ either explicitly or at least in my mind. So the second term in the derivative comes from $x^2 \cdot 2\sin(x^3)\cos(x^3)\cdot 3x^2$.

2. Using the sine difference formula,

$$A\sin(\omega t - \phi) = (A\cos\phi)\sin(\omega t) - (A\sin\phi)\cos(\omega t).$$

This is of the form $P\cos(\omega t) + Q\sin(\omega t)$ where P and Q may be set arbitrarily: Set ϕ using $\phi = \arctan(-P/Q)$ (or $\phi = \pi/2$ if $Q = 0$) and set $A = -P/\sin\phi$.

We know that $\cos(\omega t)$ and $\sin(\omega t)$ are both solutions of $x'' = -\omega^2 x$. Since differentiation is linear, $(cf + g)' = cf' + g'$, $u(t) = A\cos(\omega t) + B\sin(\omega t)$ is also a solution for any A and B, with $u(0) = A$ and $u'(0) = \omega B$.

These are the only solutions, because if $v(t)$ is a solution with $v(0) = A$ and $v'(0) = \omega B$, then $w = u - v$ is a solution of $w'' = -\omega^2 w$ with $w(0) = 0$ and $w'(0) = 0$. By the chain rule,

$$\frac{d}{dt}1/2(\omega^2 w(t)^2 + w'(t)^2) = \omega^2 w(t)w'(t) + w'(t)w''(t) = w'(t)(w(t) - \omega^2 w(t)) = 0.$$

So $1/2(\omega^2 w(t)^2 + w'(t)^2)$ is a constant, and since it is equal to zero at $t = 0$, it must equal zero for all t. And since this quantity is a sum of squares, the individual terms must then be zero too. So $w(t)^2 = 0$ so $w(t) = 0$.

By the same arguments, we can also write the general solution in the form $x(t) = B\cos(\omega t - \psi)$.

For the form $x(t) = A\sin(\omega t - \phi)$, A is called the amplitude, and gives the maximum and minimum heights of the graph, and ϕ is a phase shift: the zero of $\sin t$ at the origin is shifted to where $\omega t - \phi = 0$, i.e., where $t = \phi/\omega$.

We know this is where the slope of the sine function is at its maximum (or minimum, depending upon the sign of A) so the velocity is maximized at

$$t = \phi/\omega + n\pi/\omega, \quad n = \ldots - 1, 0, 1, \ldots$$

The displacement is maximized or minimized where the argument of the sine function is $\pi/2 + n\pi, \quad n = \ldots - 1, 0, 1, \ldots$, so setting this to $\omega t - \phi$ we get

$$t = \frac{\phi + \pi/2 + n\pi}{\omega}, \qquad n = \ldots - 1, 0, 1, \ldots$$

For $x(t) = 2\sin(t - \pi/4)$, velocity is extremized when $x(t) = 0$, or $t = n\pi + \pi/4$, position is extremized when $v(t) = x'(t) = 0$, or $t = (n + \frac{1}{2})\pi + \pi/4$. A figure is at the end of these solutions.

3. The area and the radius are related statically by $A = \pi r^2$ and in this situation they are related dynamically by $A(t) = \pi r(t)^2$. Taking derivatives, we get $A'(t) = 2\pi r(t) r'(t)$. Given $r'(t) = 1.5$ feet/second and $t = 2$ hours, we get Converting to seconds, $A'(7200) = 2\pi(1.5) r(7200)$. The problem implies that the initial radius of the slick is zero feet, so that $r(t) = 1.5t$, and $r(7200) = 10800$ and the slick is growing at 32400π square feet per second.

   Alternatively, we can directly substitute $r(t) = 1.5t$ into the area formula, $A(t) = 2.25\pi t^2$ and $A'(t) = 4.5\pi t$ so at $t = 2$ hours or 7200 seconds, $A' = 32400$.

4. Using t in hours, with the center of the face of the clock at the origin in the $x - y$ plane and units in inches, the position of the minute hand is $(m_x(t), m_y(t)) = (5\sin(2\pi t), 5\cos(2\pi t))$ and the hour hand is $(h_x(t), h_y(t)) = (4\sin(2\pi t/12), 4\cos(2\pi t/12))$ (clock parametrization behaves like the inverse function of standard circular parametrization, starting from $(0, 1)$ and going clockwise instead of from $(1, 0)$ and going counter-clockwise, hence the reversal of x and y.) (Another option is to make 3 o'clock $t = 0$,

   and then standard parameterizations of the form $(r\cos(-kt), r\sin(-kt))$ work nicely, and the first minus sign can be ignored by the evenness of cosine. It would also be nicer if the hands had length 3 and 4 instead of 4 and 5 so things would work out in rational numbers!)

   So $d(t)^2 = (m_x(t) - h_x(t))^2 + (m_y(t) - h_y(t))^2$.
   Then $2d(t)d'(t) = 2(m_x(t) - h_x(t)(m_x(t) - h_x(t))' + (m_y(t) - h_y(t))(m_y(t) - h_y(t))')$.

Plug in $t = 3$, and divide the left hand side by $2d(3)$ to get $d'(3)$. The necessary values will be $h_x(3) = 4, h_y(3) = 0$ and $m_x(3) = 0, m_y(3) = 5$, and $h'_x(3) = 0, h'_y(3) = -8\pi/12$ and $m'_x(3) = 10\pi, m'_y(3) = 0$.

So $d'(3) = \frac{-40\pi+40\pi/12}{\sqrt{41}}$ inches per hour.

To check this, we use the approximation that the hour hand is stuck at 3 o'clock, at which time the tip of the minute hand moves at 10π inches per hour to the right. The diagonal velocity is the horizontal velocity over the cosine of the angle between the hands, or $(10\pi)\frac{4}{\sqrt{41}}$. If we freeze the minute hand, the same analysis of the hour hand moving down at $8\pi/12$ inches per hour gives an opposite contribution of vertical velocity times the sine of the angle at the minute hand, or $(8\pi/12)\frac{5}{\sqrt{41}}$. Combining these contributions, we have found another probably easier route to the same answer.

5. Use $y(x + dx) \approx y(x) + dyy(x) + y'(x)dx$. $dy\frac{dy}{dx}dx$ In the first one, $y = x^{1/2}$, $x = 64$, $y = 8$, $dx = 2$, $y' = 1/2x^{-1/2}$, $y'(64) = 1/16$ so $(66)^{1/2} \approx 8\frac{2}{16}$.

   In the second one, $y = \sin x$, $x = 0$, $dx = \frac{\pi}{100}$, $y'(x) = \cos x$ $y(0) = 0$, $y'(0) = 1$, and $\sin(\frac{\pi}{100}) \approx \frac{\pi}{100}$.

   For $m(v) = m_o(1-(v/c)^2)^{-1/2}$, $m'(v) = m_o(1-(v/c)^2)^{-3/2}\frac{v}{c^2}$, and we are asked to find $100\frac{dm}{m} = 100\frac{m'(v_o)dv}{m(v_o)}$. Plugging in $v_o/c = .9$ and $dv/c = .02$ we get $100(.02c)\frac{m'(v_o)}{m(v_o)} = 2c\frac{v_o}{c^2}(1 - (v_o/c)^2)^{-1}$. Simplifying, the percent increase is $\frac{2(.9)}{(1-.81)} \approx 9.5\%$.

6. Similar to the previous problems, with $y(r) = 4/3\pi r^3$, $y'(r) = 4\pi r^2$ and $y(3) = 36\pi$, and coincidentally, $y'(3) = 36\pi$ and $dy = 36\pi(.025)$ cubic inches.

7. Consider $f(x) = \begin{cases} |x|, x < 0, \\ \sin x, x \geq 0. \end{cases}$ Find all local maxima and minima of f, where f is increasing and decreasing, where f is concave up and concave down, and all inflection points. Does f have a global maximum or a global minimum? Sketch the graph of $f(x)$.

   For $x < 0$, $f(x) = |x| = -x$ so $f'(x) = -1$ and f is decreasing. For $x > 0$, $f(x) = \sin x$, so $f'(x) = \cos x$. So $f' > 0$ and f is increasing when $0 < x < \pi/2$ and when $3\pi/2 + 2\pi n < x < 5\pi/2 + 2\pi n, \quad n = 0, 1, \ldots$ and $f' < 0$ and f is decreasing when $\pi/2 + 2\pi n < x < 3\pi/2 + 2\pi n, \quad n = 0, 1, \ldots$. The local minima are where f changes from decreasing to increasing, at $x = 0$ and at $x = 3\pi/2 + 2\pi n, \quad n = 0, 1, \ldots$, and the local maxima are where f changes from increasing to decreasing, at $x = 0$ and at $x = \pi/2 + 2\pi n, \quad n = 0, 1, \ldots$. For $x < 0$, $f''(x) = 0$, and for $x > 0$, $f''(x) = -\sin x$, so f is concave up when $\sin x$ is negative, i.e., when $\pi + 2\pi n < x < 2\pi + 2\pi n, \quad n = 0, 1, \ldots$ and f is concave down when $\sin x$ is positive, i.e., when $2\pi n < x < \pi + 2\pi n, \quad n = 0, 1, \ldots$ There is no global maximum since $f(x) \to \infty$ as $x \to -\infty$. The global minimum value of $f(x) = -1$ is taken at all of the local minima with $x > 0$, at $x = 3\pi/2 + 2\pi n, \quad n = 0, 1, \ldots$ There are inflection points where $f'' = 0$ and f changes concavity, at $x = n\pi, \quad n = 1, 2, \ldots$

   For $f(x) = x^3 - 12x + 1$, $f'(x) = 3x^2 - 12$ and $f''(x) = 6x$. Then $f' = 0$ when $x^2 = 4$ or $x = \pm 2$. For $x < -2$, f is increasing, f is decreasing for $-2 < x < 2$ and f is increasing for $x > 2$. So f has a local maximum at $x = -2$ and a local minimum at $x = 2$. There is no global maximum or minimum. f is concave up where $f'' > 0$, i.e., for $x > 0$, and concave down for $x < 0$. There is one inflection point where f changes concavity, at $x = 0$ where $f''(x) = 0$. See Sec. 4.2, #19.

For $f(x) = \dfrac{1}{1+x^2}$, $f'(x) = \frac{-2x}{(1+x^2)^2}$. So f is increasing where $f' > 0$, i.e., for $x < 0$ (the denominator of f' is always positive) and f is decreasing where $f' < 0$, i.e., for $x > 0$. There is one local maximum at $x = 0$ where f changes from increasing to decreasing. This is also a global maximum. There are no local or global minima as f approaches zero but remains positive as x tends to infinity in both directions. By the quotient rule, $f''(x) = \frac{6x^2-2}{(1+x^2)^3}$, so $f'' = 0$ and f has inflection points when $x = \pm\frac{\sqrt{3}}{3}$, $f'' < 0$ and f is concave down between these points, and $f'' > 0$ and f is concave up outside these points. See example 4, p. 170, section 4.2 for a similar problem.

8. $f'(x) = 2\cos(x - \frac{\pi}{4})$ which is positive for $\frac{\pi}{2} < x < \frac{3\pi}{4}$ (where f is increasing) and negative for $\frac{3\pi}{4} < x < \frac{5\pi}{4}$ (where f is decreasing.) There is a local maximum at $x = \frac{3\pi}{4}$ where $f = 2$, $f' = 0$, and $f'' < 0$, and no other critical points in the interval. Comparing with the endpoints $f(\frac{\pi}{2}) = \sqrt{2}$ and $f(\frac{5\pi}{4}) = 0$, the global maximum is 2 and the global minimum is 0 on this interval.

9. Since the corners are $(0, x), (0, -x), (x, 12 - x^2), (-x, 12 - x^2)$ the area will be $A = 2xy = 2x(12 - x^2) = -2x^3 + 24x$. $\dfrac{dA}{dx} = 2(12 - 3x^2) = 0$ which is zero when $x = \pm 2$. We take x to be the vertex on the right, so $x = 2, y = 8$, the dimensions are 4 by 8 (and the maximum area is 32.)

10. The total travel time $T(x)$ is the sum of the times in the two media. Using velocity = distance/time, we get $t_1 = d_1/c_1$ and $t_2 = d_2/c_2$. We minimize $T(x) = t_1 + t_2 = d_1(x)/c_1 + d_2(x)/c_2$, by setting $T' = 0 = \frac{d_1'}{c_1} + \frac{d_2'}{c_2}$ or $\frac{d_1'}{c_1} = \frac{-d_2'}{c_2}$. By Pythagoras and the diagram, $d_1^2 = x^2 + a^2$ and $d_2^2 = (d-x)^2 + b^2$, so implicit differentiation gives $2d_1 d_1' = 2x$ and $2d_2 d_2' = -2(d-x)$. So $d_1' = \frac{x}{d_1}$ and $d_2' = -\frac{d-x}{d_2}$. From the diagram, $\frac{x}{d_1} = \sin\theta_1$ and $\frac{d-x}{d_2} = \sin\theta_2$. Putting this all together, we get Snell's Law: $\frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2}$.

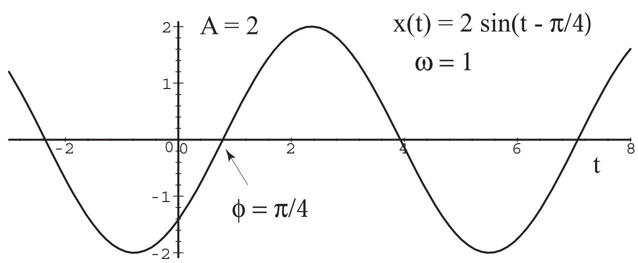

Figure 1: Graph of $f(x)$ for #2.