Mathematics 1220 PRACTICE EXAM IV Spring 2006

1. Solve the following differential equations.

   (a) $y'' + y = 0$ (b) $y'' + 5y' - 6y = 0$

   (c) $y'' - 4y' + 4y = 0$ (d) $y'' + 4y = 0, \quad y(0) = 0$ and $y'(0) = 1$

   (e) $y'' + y' + y = 0$ (f) $y'' + y' - 6y = 2x^2$ (g) $y'' + 4y' = \cos x$

2. Consider a mass m on a spring with constant k. The position $x(t)$ of the mass at time t satisfies the second order constant coefficient differential equation $m\dfrac{d^2x}{dt^2} = -kx$, or

$$\frac{d^2x}{dt^2} + \omega^2 x = 0, \quad \omega = \sqrt{\frac{k}{m}}\,. \tag{1}$$

   Use the characteristic polynomial of (1) to find its general solution, and then from your general solution find the particular solution with initial position $x(0) = 0$ and initial velocity $\dfrac{dx}{dt}(0) = 1$.

3. Consider a microwave (or radar wave) propagating vertically with electric field in the $x-$direction at time t given by $E(x,t)$. Assuming the wave is time harmonic with one angular frequency ω, then $E(x,t) = e^{i\omega t}E(x)$. If the wave propagates through a medium (such as a turkey or the atmosphere) described by the permittivity, or dielectric constant, $\epsilon(x)$ (related to the index of refraction $n(x)$ by $n = \sqrt{\epsilon}$), then $E(x)$ satisfies the second order non-constant coefficient differential equation

$$\frac{d^2E}{dx^2} + k_0^2\epsilon(x)E(x) = 0,$$

   where $k_0 = \omega/c$ is the free space wave number, and c is the speed of light. For a homogeneous medium, such as free space where $\epsilon(x) = 1$, E satisfies

$$\frac{d^2E}{dx^2} + k_0^2 E = 0. \tag{2}$$

   Use the characteristic polynomial of (2) to find its general solution, and then from your general solution find the particular solution satisfying $E(0) = 0$ and $\dfrac{dE}{dx}(0) = 1$.

4. Consider a pendulum of length ℓ. The angle $\theta(t)$ that the pendulum makes with respect to vertical at time t satisfies the second order nonlinear differential equation $\ell\dfrac{d^2\theta}{dt^2} = -g\sin\theta$, where g is the acceleration due to gravity. For small oscillations (small θ), $\sin\theta$ can be approximated by θ, which yields the linearized equation

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0, \quad \omega = \sqrt{\frac{g}{\ell}}\,. \tag{3}$$

   Use the characteristic polynomial of (3) to find its general solution, and then from your general solution find the particular solution with initial angle $\theta(0) = \pi/16$ and initial angular velocity $\dfrac{d\theta}{dt}(0) = 0$.