

**Mathematics 1220      PRACTICE EXAM II      Spring 2009**  
**ANSWERS**

1. (a) Let  $x = 1/n$ . Then  $\lim_{x \rightarrow 0} \frac{x^{-x} - 1}{x} = -\lim_{x \rightarrow 0} x^{-x}(\ln x + 1)$  diverges,  $\rightarrow +\infty$ .  
 (b) Let  $y = (\cos x)^{\csc x}$ , and apply L'Hôpital's rule to  $\ln y$ , giving  $y \rightarrow 1$ .  
 (c) Apply L'Hôpital's rule 25 times to  $x^{25}/e^x$ , giving 0.  
 (d) Apply L'Hôpital's rule:  $\lim_{x \rightarrow \infty} \frac{x e^x}{e^{x^2/2}} = \lim_{x \rightarrow \infty} e^{(-\frac{x^2}{2} + x)} \left(1 + \frac{1}{x}\right)$ . Since  $-\frac{x^2}{2} + x = \frac{x}{2}(2 - x) < 0$  for  $x > 2$ , the limit is 0.  
 (e)  $\lim_{x \rightarrow -\infty} (e^{-x} - x) = \lim_{x \rightarrow +\infty} (e^x + x) \rightarrow +\infty$ .  
 (f) Apply L'Hôpital's rule, and use the Second Fundamental Theorem of Calculus on the numerator, which gives  $\sin 1$  as the limit.  
 (g) Apply L'Hôpital's rule, and use the Second Fundamental Theorem of Calculus on the numerator. Another application of L'Hôpital's rule gives  $1/3$ .  
 (h) 3      (i)  $-1/2$
  
2. (a) 0 for  $m = n$  and  $m \neq n$ , use  $\sin mx \cos nx = \frac{1}{2} [\sin(m+n)x + \sin(m-n)x]$ .  
 (b)  $\pi$ , use  $\sin mx \sin nx = -\frac{1}{2} [\cos 2mx - 1]$  for  $m = n$ .  
 (c)  $\frac{u^2}{2} + C$ ,  $u = \ln(\cosh x)$ .      (d)  $\frac{1}{2} \tan^{-1}\left(\frac{u+1}{2}\right) + C$ ,  $u = e^x$ .  
 (e)  $\int \frac{3x-1}{x^2-4} dx = \int \left(\frac{7}{4} \frac{1}{x+2} + \frac{5}{4} \frac{1}{x-2}\right) dx = \frac{7}{4} \ln|x+2| + \frac{5}{4} \ln|x-2| + C$   
 (f)  $\frac{x}{2} [\cos(\ln x) + \sin(\ln x)]$ , use two integration by parts, the first with  $u = \cos(\ln x)$ ,  $dv = dx$ , to solve for the integral.  
 (g)  $\frac{1}{2} (A \ln|x| + B \ln|x+3|) + C$ ,  $\frac{1}{x(x+3)} = \frac{A}{x} + \frac{B}{x+3}$ ,  $A = 1/3$ ,  $B = -1/3$ .  
 (h)  $-\frac{e^u}{2} + C$ ,  $u = t^2 + 2t + 5$ .  
 (i)  $\frac{1}{3} \cos^3 x - \cos x + C$ , use the identity  $\sin^2 x + \cos^2 x = 1$ .  
 (j)  $x e^x - e^x + C$ , do integration by parts with  $u = x$  and  $dv = e^x dx$ .  
 (k)  $A \ln|x+5| + B \ln|x-2| + C$ ,  $\frac{3x-13}{x^2+3x-10} = \frac{A}{x+5} + \frac{B}{x-2}$ ,  $A = 2$ ,  $B = 1$ .
  
3. (a) Let  $u = \ln x$ , then  $p = 1/2$  for infinite domain  $\Rightarrow$  divergence.  
 (b) Bound the integrand above with  $C/x^p$ , for any  $p$  such that  $1 < p < 3/2$ , like  $p = 5/4 \Rightarrow$  convergence.  
 (c)  $|e^{-x} \cos x| \leq e^{-x} \Rightarrow$  convergence.  
 (d) Near  $x = 0$ ,  $(e^{-x^2}/x^2) \sim (1/x^2) \Rightarrow$  divergence.  
 (e) 16,000 integration by parts reduces the integral to a purely decaying exponential  $\Rightarrow$  convergence; or use comparison  $x^{16,000} e^{-x} < e^{-x/2}$  for sufficiently large  $x$ , but you must show this!

- (f) Near  $x = 0$ , integrand  $\sim 1/x^{2/3} \Rightarrow$  convergence.
4. (a) diverges ( $n^{\text{th}}$  term test, or geometric series with  $|r| > 1$ )  
 (b) converges absolutely (telescopes, or p-test)  
 (c) diverges  $\frac{1}{k+2} \sim \frac{1}{k}$   
 (d) diverges  $\sqrt{n} \sin\left(\frac{1}{n}\right) \sim \frac{1}{\sqrt{n}}$   
 (e) diverges ( $n^{\text{th}}$  term test)  
 (f) converges absolutely (integral test)  
 (g)  $\sqrt{1 - \cos\left(\frac{1}{n}\right)} \sim \frac{1}{n} \Rightarrow$  divergence
5. 4 meters
6. (a) diverges, as  $\lim_{2n \rightarrow \infty} (-1)^{2n} \frac{2n}{2n+2} = 1$ , but  $\lim_{2n+1 \rightarrow \infty} (-1)^{2n+1} \frac{2n+1}{(2n+1)+2} = -1$ .  
 (b) converges to  $\frac{1}{2}$  as  $\frac{\sqrt{n^2+4}}{2n+1} \sim \frac{n}{2n} = \frac{1}{2}$ .  
 (c) converges to 0. Apply the squeeze theorem to  $\frac{-1}{n^{1/2}} \leq \frac{\cos(2n)}{n^{1/2}} \leq \frac{1}{n^{1/2}}$ .  
 (d) converges to 0 as  $\ln\left(\frac{n}{n+1}\right) \sim \ln\left(\frac{n}{n}\right) = 0$ .