Mathematics 1220 PRACTICE EXAM III Spring 2009

ANSWERS

1. $\ln(1+x) = x - x^2/2 + x^3/3 - x^4/4 + ...$, radius $= 1$
   $1/(1+x)^2 = 1 - 2x + 3x^2 - 4x^3 + ...$, radius $= 1$

2. $\frac{1}{4+x^2} = \frac{1}{4}\frac{1}{1+\left(\frac{x}{2}\right)^2} = \frac{1}{4}\left(1 - \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^4 + \ldots\right)$, radius $= 2$

3. $\sin^2 x = \frac{1}{2}\left(\frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} + \frac{(2x)^6}{6!} - \ldots\right)$, infinite radius

4. $\cosh x = 1 + x^2/2! + x^4/4! + x^6/6! + ...$, infinite radius

5. (a) $(-1, 1/3)$, conditionally convergent at -1, divergent at $1/3$
   (b) $(-1/\tau, 1/\tau)$, $\tau = (1+\sqrt{5})/2$

6. (a) $|e^{-n}\sin n| \le e^{-n} \longrightarrow 0$
   (b) 1
   (c) Does not exist due to oscillation
   (d) $\int_0^1 e^{x^2}\,dx = 1 + \frac{1}{3} + \frac{1}{5\cdot 2!} + \frac{1}{7\cdot 3!} + ...$

7. 4 meters

8. (a) converges conditionally
   (b) $\sqrt{1-\cos\left(\frac{1}{n}\right)} \sim \frac{1}{n} \Longrightarrow$ divergence
   (c) converges absolutely (ratio test)
   (d) diverges (n^{th} term test)
   (e) converges absolutely (integral test)
   (f) converges absolutely (ratio test)

9. $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$, $\frac{d^2y}{dx^2} = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \ldots$, and we have $\frac{d^2y}{dx^2} = -y$ and the initial conditions $y(0) = a_0 = 0$ and $y'(0) = a_1 = 1$, so we find

$$a_2 = \frac{-a_0}{2} = 0 \quad a_3 = \frac{-a_1}{6} = \frac{-1}{3!}$$
$$a_4 = \frac{-a_2}{12} = 0 \quad a_5 = \frac{-a_3}{20} = \frac{1}{5!}$$

   and so on, so $y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sin x$.

10. Solve each of the following ODEs. If the initial conditions are given, use them to find the constants in the general solution.

    (a)
$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 2y = 0$$
    Characteristic Equation: $r^2 + 4r - 2 = 0$
    Solution of Characteristic Equation: $r = -2 \pm \sqrt{6}$
    General Solution to ODE:
$$y(t) = c_1 e^{(-2-\sqrt{6})t} + c_2 e^{(-2+\sqrt{6})t}$$

(b)

$$2\frac{d^2y}{dt^2} - 3\frac{dy}{dt} - 2y = 0, \qquad y(0) = 1, y'(0) = 2$$

Characteristic Equation: $2r^2 - 3r - 2 = (2r-1)(r+2) = 0$
Solution of Characteristic Equation: $r = -2, \frac{1}{2}$
General Solution to ODE:

$$y(t) = c_1 e^{-2t} + c_2 e^{\frac{1}{2}t}$$

Solving for c_1 and c_2:

$$\begin{aligned} y(0) = 1 &= c_1 + c_2 \\ y'(0) = 2 &= -2c_1 + \frac{1}{2}c_2 \end{aligned}$$

$\rightarrow c_1 = -\frac{3}{5}, c_2 = \frac{8}{5}$
Solution to ODE:

$$y(t) = -\frac{3}{5}e^{-2t} + \frac{8}{5}e^{\frac{1}{2}t}$$

(c)

$$2\frac{d^2y}{dt^2} + 5\frac{dy}{dt}y = 0, \qquad y(0) = 0, y'(0) = -1$$

Characteristic Equation: $2r^2 + 5r = r(2r+5) = 0$
Solution of Characteristic Equation: $r = 0, -\frac{5}{2}$
General Solution to ODE:

$$y(t) = c_1 + c_2 e^{-\frac{5}{2}t}$$

Solving for c_1 and c_2:

$$\begin{aligned} y(0) = 0 &= c_1 + c_2 \\ y'(0) = -1 &= -\frac{5}{2}c_2 \end{aligned}$$

$\rightarrow c_1 = -\frac{2}{5}, c_2 = \frac{2}{5}$
Solution to the ODE:

$$y(t) = -\frac{2}{5} + \frac{2}{5}e^{-\frac{5}{2}t}$$