EXAM II SOLUTIONS

1. (a) $\displaystyle\int \frac{dx}{x^2+6x+10} = \int \frac{dx}{(x+3)^2+1} = \tan^{-1}(x+3) + C.$

   (b)
$$\begin{aligned}\int \frac{2x+1}{x^2-1}\,dx &= \int \frac{1}{2(x+1)}\,dx + \int \frac{3}{2(x-1)}\,dx\\ &= \frac{1}{2}\ln(x+1) + \frac{3}{2}\ln(x-1) + C.\end{aligned}$$

   (c) Let $u = \tan^{-1} 2x$, then $du = \frac{2}{1+4x^2}\,dx$ so
$$\int \frac{e^{\tan^{-1}2x}}{1+4x^2}\,dx = \int \frac{1}{2}e^u\,du = \frac{1}{2}e^u = \frac{1}{2}e^{\tan^{-1}2x}.$$

2. $\displaystyle\int_0^b xe^{-x}\,dx = -be^{-b} - e^{-b} + 1$ and so
$$\int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty} -be^{-b} - e^{-b} + 1 = 1.$$

3. (a) $\displaystyle\int_{-1}^1 \frac{1}{x^2}\,dx = 2\int_0^1 \frac{1}{x^2}\,dx$ diverges since the power of the x is greater than 1.

   (b) The function $\frac{1}{x(1-x)}$ goes to 0 as x goes to infinity like $\frac{1}{x^2}$, but goes to infinity as x goes to 1 like $\frac{1}{1-x}$, therefore the integral diverges.
$\displaystyle\int_1^\infty \frac{dx}{x(1-x)} = \int_1^\infty \frac{1}{x} + \frac{1}{1-x}\,dx = \ln\left(\frac{x}{1-x}\right)\Big|_1^\infty = \lim_{x\to\infty} \ln\left(\frac{x}{1-x}\right) - \lim_{x\to 1} \ln\left(\frac{x}{1-x}\right) = -\infty$

   (c) $\displaystyle\int \sin x\,dx = \cos x$ and $\displaystyle\lim_{x\to\infty} \cos x$ does not exist so the integral diverges.

4. (a) By L'Hopital's rule,
$$\lim_{x\to 1} \frac{\ln x - x + 1}{x^3 - 3x + 2} = \lim_{x\to 1} \frac{\frac{1}{x} - 1}{3x^2 - 3} = \lim_{x\to 1} \frac{-1/x^2}{6x} = -\frac{1}{6}.$$

   (b) $\displaystyle\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x}$. We apply L'Hopital's rule to
$$\lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0.$$

   Therefore, $\displaystyle\lim_{x\to 0^+} x^x = e^0 = 1.$

(c) By L'Hopital's rule,

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = \lim_{x\to 0} \frac{\cos x - 1}{3x^2} = \lim_{x\to 0} \frac{-\sin x}{6x} = \lim_{x\to 0} \frac{-\cos x}{6} = -\frac{1}{6}.$$

5. (a)

$$\lim_{n\to\infty} \frac{2n^2 + 3n + 1}{n^2 + 2} = \lim_{x\to\infty} \frac{2x^2 + 3x + 1}{n^2 + 2} = \lim_{x\to\infty} \frac{4x + 3}{2x} = \lim_{x\to\infty} \frac{4}{2} = 2.$$

(b) Since for all n, $-1 \leq \sin n \leq 1$, we have

$$\lim_{n\to\infty} -\frac{1}{n} \leq \lim_{n\to\infty} \frac{\sin n}{n} \leq \lim_{n\to\infty} \frac{1}{n}$$

so by the squeeze theorem, we find $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

6. (a) If we let $n = k + 1$, then $\sum_{k=1}^{\infty} \left(\frac{1}{k+1} \right) = \sum_{k=2}^{\infty} \frac{1}{n} = \infty$ since the harmonic series diverges.

(b) This question can be solved by recognizing the geometric series. $\sum_{n=0}^{\infty} e^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{e} \right)^n = \frac{1}{1 - \frac{1}{e}}$