Mathematics 1220 EXAM I Spring 2009

ANSWERS

1. a. By chain rule, $\frac{d}{dx}\left(e^{\sqrt{x}}\right) = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$.

   b. The derivative of $\tan^{-1} x$ is $\frac{1}{1+x^2}$, so the derivative of $\tan^{-1}(x^2)$ is $\frac{2x}{1+x^4}$, by the chain rule.

   c. Since $\lim_{x\to 0}(1+x)^{1/x} = e$, we have $\lim_{x\to 0}(1+ex)^{1/x} = \left[\lim_{ex\to 0}(1+ex)^{1/ex}\right]^e = e^e$.

2. a. Let $u = 1+\sqrt{x}$, then $2du = \frac{dx}{\sqrt{x}}$ and so

$$\int \frac{1}{\sqrt{x}(1+\sqrt{x})}\,dx = \int \frac{2du}{u} = 2\ln u + C = 2\ln(1+\sqrt{x}) + C.$$

   b. Let $u = 2^x$, then $du = \ln(2)2^x dx$ and so

$$\int 2^x \sinh 2^x\,dx = \int \frac{\sinh u}{\ln 2}\,du = \frac{\cosh u}{\ln 2} + C = \frac{\cosh 2^x}{\ln 2} + C.$$

   c. Use integration by parts with $u = lnx$ and $dv = x^3 dx$. Hence $du = \frac{dx}{x}$, $v = \frac{x^4}{4}$, and then $\int x^3\ln(x)dx = \frac{x^4}{4}lnx - \int \frac{x^3}{4}dx = \frac{x^4}{4}lnx - \frac{x^4}{16} + C$.

   d. Use integration by parts twice. Let $u = x^2$ and $dv = e^x$. Then $du = 2xdx$, $v = e^x$, and $\int x^2e^x dx = x^2e^x - 2\int xe^x dx$. Let $u = x$ and $dv = e^x dx$ for the second term.. Then $du = dx$, $v = e^x$, and $\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x$. In summary, $\int x^2e^x dx = x^2e^x - 2\int xe^x dx = x^2e^x - 2(xe^x - e^x) + C$.

   e. Let $u = sin(x)$, then $du = cos(x)dx$. As $cos^2(x) = 1 - sin^2(x) = 1 - u^2$, $\int \cos^3(x)dx = \int(1-u^2)du = u - \frac{u^3}{3} + C = sin(x) - \frac{sin^3(x)}{3} + C$.

3. The growth of the bacteria population is modelled by the equation $P(t) = 100e^{kt}$, where t is in hours. We are given $P(0.75) = 200 = 100e^{\frac{3}{4}k}$, so we can solve for k to get $k = \frac{4}{3}\ln 2$. Therefore, $P(t) = 100\left(2^{\frac{4}{3}t}\right)$, so after $t = 3$ hours, $P(3) = 1600$.

4. You should put in at least $A_0 = 1000000(1.1)^{-50} \approx \$8519$.

5. Our integrating factor is $e^{\int \tan x\,dx} = \sec x$. Then, $\frac{d}{dx}(y\sec x) = 2x$ and so $y\sec x = x^2 + C$. Since $y(0) = 0$, we see $C = 0$. Therefore, $y(x) = x^2\cos x$.

6. We are given $k = 0.5, T = 40$ and $\theta(0) = 60$ (assuming that $t = 0$ at 6am). After making the substitutions into the ODE and writing it in standard form gives:

$$\frac{d\theta}{dt} + 0.5\theta \quad = \quad +20$$

   Using the integrating factor $e^{0.5\int dt} = e^{0.5t}$, we get:

$$\frac{d}{dt}(e^{0.5t}\theta) \quad = \quad 20e^{0.5t} \quad \rightarrow \quad e^{0.5t}\theta = 40e^{0.5t} + C$$

Using $\theta(0) = 60$, we see that $C = 20$, so the full solution to the DE is:

$$\theta(t) = 40 + 20e^{-0.5t}$$

We now want to find out when the corpse last had a body temperature of 98.6°F.

$$\begin{aligned} \theta(t_d) = 98.6 &= 40 + 20e^{-0.5t} \\ \rightarrow \quad t_d &= -2\ln\left(\frac{58.6}{20}\right) \approx -2.15 \approx -2hrs\ 9min \end{aligned}$$

Therefore, the person died at 6am − 2 hrs and 9 min = 3:51am.